Discussion of vector finite element solution of the TM cutoff modes of a waveguide.

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1 Introdction

This document serves to describe the formulation used to solve the TM cutoff modes of a rectangular waveguide using the magnetic field formulation.

2 Formulation

Inside a closed waveguide the magnetic field, $\mathbf{H}$, satisfies the following vector differential equation

$$\nabla \times \left(\frac{1}{\epsilon_r} \nabla \times \mathbf{H} \right) - k_0^2 \mu_r \mathbf{H}, \tag{1}$$

with the following boundary conditions

$$\hat{n} \times (\nabla \times \mathbf{H}) = 0 \quad \text{on } \Gamma_e, \tag{2}$$

$$\hat{n} \times \mathbf{H} = 0 \quad \text{on } \Gamma_m. \tag{3}$$

Here Γ_e and Γ_m are electric and magnetic walls bounding the domain Ω respecitvely. This results in the following functional

$$F(\mathbf{H}) = \frac{1}{2} \int\int_\Omega \left[\frac{1}{\epsilon_r} (\nabla \times \mathbf{H}) \cdot (\nabla \times \mathbf{H})^* - k_0^2 \mu_r \mathbf{H} \cdot \mathbf{H}^* \right] d\Omega. \tag{4}$$

By breaking the magnetic field into a longitudinal (z) and transverse (x and y) components, it can be shown that[1] at cutoff for the TM mode of a hollow waveguide ($\epsilon_r = 1$, $\mu_r = 1$) this functional can be written as

$$F(\mathbf{H}) = \frac{1}{2} \int\int_\Omega \left[(\nabla_t \times \mathbf{H}_t) \cdot (\nabla_t \times \mathbf{H}_t)^* - k_0^2 \mathbf{H}_t \cdot \mathbf{H}_t^* \right] d\Omega, \tag{5}$$

with $\mathbf{H}_t$ indicating the transverse components of the magnetic field and $\nabla_t \times$ the transverse curl operator.

[1] Still to come.

After discretisation, the minimization of this functional reduces to solving the following generalized eigenvalue equation

$$[S]H_t = k_0^2[T]H_t, \tag{6}$$

the solution of which results in finding the cutoff wavenumber, k_0, and the field distributions for a given cutoff mode.

Note that the physical solutions of k_0 are all non-zero, and thus the zero eigenvalues (so called spurious modes) must be excluded.